Event-triggered stabilization over digital channels of linear systems with disturbances ⋆

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Abstract

In the same way that subsequent pauses in spoken language are used to convey information, it is also possible to transmit information in communication systems not only by message content (data payload), but also with its timing. This paper presents an event-triggering strategy that utilizes timing information by transmitting in a state-dependent fashion. We consider the stabilization of a continuous-time, time-invariant, linear plant over a digital communication channel with bounded delay and subject to bounded plant disturbances. We propose an encoding-decoding scheme that guarantees a sufficient information transmission rate for stabilization of the plant. We also determine a lower bound on the information transmission rate of the sensor, which is necessary for stabilization. For small values of the delay, we show that the timing information *implicit* in the triggering events is enough to stabilize the plant with any positive information transmission rate. In contrast, when the delay increases beyond a critical threshold, the timing information alone is not enough, and the data payload transmission rate begins to increase. Large values of the delay require transmission rates higher than what the classic *data-rate theorem* prescribes. Our results also provide a novel encoding-decoding scheme for complex systems, which can be readily applied to diagonalizable multivariate systems with complex eigenvalues. The identified rates depend on parameters in the event-triggered law and plant dynamics, and the bounds on the plant disturbances and channel delays. The technical treatment does not rely upon any a priori probabilistic distribution of plant disturbances, initial condition, or delay.

Key words: Control under communication constraints; event-triggered control; networked control systems; stabilization under delays.

1 Introduction

For many cyber-physical systems, the feedback loop is closed over a communication channel [Kim and Kumar, 2012]. In this context, data-rate theorems state that the minimum communication rate to achieve stabilization is equal to the *entropy rate* of the plant, expressed by the sum of the unstable modes in nats (one nat corresponds to $1/\ln 2$ bits.) Key contributions by Tatikonda and Mitter [2004a], Nair and Evans [2004], and Liberzon [2003] consider a "bit-pipe" communication channel, capable of noiseless transmission of a finite number of bits per unit time evolution of the plant. Extensions to noisy communication channels are considered in [Sahai and Mitter, 2006, Matveev and Savkin, 2009, Yüksel and Başar, 2013]. Stabilization over time-varying bit-pipe channels, including the erasure channel as a special case, are studied by Minero et al. [2009]. Additional formulations include stabilization of switched linear systems [Liberzon, 2014], uncertain systems [Ishii, 2010], multiplicative noise [Ding et al., 2018], optimal control [Kostina and Hassibi, 2019, Khina et al., 2019], and stabilization using event-triggered strategies [Tallapragada and Cortés, 2016, Pearson et al., 2017, Linsenmayer et al., 2017, Tallapragada et al., 2018, Demirel et al., 2017, Li et al., 2012].

While the majority of communication systems transmit information by adjusting the content of the message, it is also possible to communicate information by adjusting the transmission time of a symbol [Anantharam and Verdú, 1996]. The work [Khojasteh et al., 2018a] studies the fundamental limitations of using timing information for stabilization and show that it is possible to stabilize a plant using inherent information in the timing of the transmissions. In fact, it is known that event-triggering control techniques encode information in the timing in a state-dependent fashion. The work Kofman and Braslavsky [2006] shows that, in the absence of delay in the communication process, without plant disturbances, and assuming the controller has knowledge of the triggering strategy, one can stabilize the plant with any positive data payload transmission rate. Building upon this observation, our previous work [Khojasteh et al., 2019] considers transmission delays in the communication channel and quantifies the information contained in the timing of the triggering events for the stabilization of scalar plants without disturbances. For small values of the delay, we show that stability can be achieved with any positive information transmission rate (the rate at which sensor transmits data payload). However, as the delay increases to values larger than a critical threshold, the timing information contained in the triggering action itself may not be enough to stabilize the plant and the information transmission rate must be increased. The work in [Khojasteh et al., 2019] also extends the treatment to the vector case, but the analysis is limited to plants with only real eigenvalues of the open-loop gain matrix. Furthermore, the required exponential convergence guar-

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antees lead to a mismatch between sensor and controller about the possible values of the state estimation error, which requires an additional layer of complexity in the sensor's transmission policy of the event-triggered control design. In contrast, in this work we consider the weaker stability notion of input-to-state practical stability (ISpS) Jiang et al. [1994], Sharon and Liberzon [2012], and this allows us to simplify the treatment and design a simpler event-triggered control strategy. The literature has not considered to what extent the implicit timing information in the triggering events is still useful in the presence of plant disturbances. Beyond the uncertainty due to the unknown delay in communication, disturbances add an additional degree of uncertainty to the state estimation process, whose effect needs to be properly accounted for. With this in mind, we study ISpS of a linear, time-invariant plant subject to bounded disturbance over a communication channel with bounded delay. Finally, we point out that the work Ling [2017] utilizes event-triggering to provide a sufficient bit rate condition for scalar linear systems with sufficiently small delays. The work [Ling, 2018] extends these results to second-order systems with real eigenvalues. In addition, the works [Tanwani and Teel, 2017, Tanwani et al., 2016] investigate event-triggered stabilization of linear and nonlinear systems under communication constraints, but do not explicitly quantify the effect of quantization in the presence of system disturbances or the timing information carried by the triggering events.

Our contributions are threefold. First, for scalar real plants with disturbances, we derive a sufficient condition on the information transmission rate for the whole spectrum of possible communication delay values. Specifically, we design an encoding-decoding scheme that, together with the proposed event-triggering strategy, rules out Zeno behavior and ensures that there exists a control policy which renders the plant ISpS. We show that for small values of the delay, our event-triggering strategy achieves ISpS using only implicit timing information and transmitting data payload at a rate arbitrarily close to zero. On the other hand, since larger values of the delay imply that the information transmitted has become excessively outdated and corrupted by the disturbance, increasingly higher communication rates are required as the delay becomes larger. Our second contribution pertains to the generalization of the sufficient condition to complex plants with complex open-loop gain subject to disturbances. This result sets the basis for the generalization of event-triggered control strategies that meet the bounds on the information transmission rate for the ISpS of vector systems under disturbances and with any real open-loop gain matrix (with complex eigenvalues). The first two contributions provide stronger results than our preliminary conference papers Khojasteh et al. [2018c,b] and contain a complete technical treatment. Our final contribution is a necessary condition on the information transmission rate for scalar real plants, assuming that at each triggering time the sensor transmits the smallest possible packet size to achieve the triggering goal for all realizations of the delay and plant disturbance [1] .

2 Problem formulation

We consider a networked control system described by a plant-sensor-channel-controller tuple, cf. Figure 1. The plant is de-

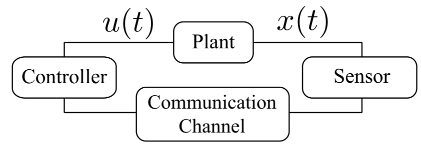

Fig. 1. System model.

scribed by a scalar, continuous-time, linear time-invariant model,

$$\dot{x} = Ax(t) + Bu(t) + w(t), \tag{1}$$

where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ for $t \in [0, \infty)$ are the plant state and control input, respectively, and $w(t) \in \mathbb{R}$ represents the plant disturbance. The latter is a Lebesgue-measurable function of time, and *upper bounded* as

$$|w(t)| \leq M, \tag{2}$$

where $M \in \mathbb{R}_{\geq 0}$. In (1), $A \in \mathbb{R}$ is positive (i.e., the plant is unstable), $B \in \mathbb{R} \setminus \{0\}$, and the initial condition $x(0)$ is bounded. We assume the sensor measurements are exact and there is no delay in the control action, which is executed with infinite precision. However, measurements are transmitted from sensor to controller over a communication channel subject to a finite data rate and bounded unknown delay. We denote by $\{t_s^k\}_{k\in\mathbb{N}}$ the sequence of times when the sensor transmits a packet of length $g(t_s^k)$ bits containing a quantized version of the encoded state. We let $\Delta'_k = t_s^{k+1} - t_s^k$ be the k^{th} *triggering interval*. The packets are delivered to the controller without error and entirely but with unknown upper bounded delay. Let $\{t_c^k\}_{k\in\mathbb{N}}$ be the sequence of times where the controller receives the packets transmitted at times $\{t_s^k\}_{k\in\mathbb{N}}$. We assume the *communication delays* $\Delta_k = t_c^k - t_s^k$, for all $k \in \mathbb{N}$, satisfy

$$\Delta_k \leq \gamma, \tag{3}$$

where $\gamma \in \mathbb{R}_{\geq 0}$. When referring to a generic triggering or reception time, for convenience we skip the super-script k in t_s^k and t_c^k, and the sub-script k in Δ_k and Δ'_k.

Remark 1 In our model clocks are synchronized at the sensor and the controller. In case of using a time stamp, due to the

[1] Throughout the paper, we use the following notation. $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{C}$, and $\mathbb{N}$ represent the set of real, nonnegative real, complex, and natural numbers, resp. We let $|.|$ and $\|.\|$ denote absolute value and complex absolute value, resp. Let log and ln represent base 2 and natural logarithms, resp. For a function $f : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$, we let $f(t^+) = \lim_{s\to t^+} f(s)$ denote the right-hand limit of f at t. In addition, $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the nearest integer less (resp. greater) than or equal to x. We denote the modulo function by $\mathrm{mod}(x, y)$, representing the remainder after division of x by y. The function $\mathrm{sign}(x)$ denotes the sign of x. Any $Q \in \mathbb{C}$ can be written as $Q = \mathrm{Re}(Q) + i\,\mathrm{Im}(Q) = \|Q\|e^{i\phi_Q}$, and for any $y \in \mathbb{R}$ we have $\|e^{Qy}\| = e^{\mathrm{Re}(Q)y}$. $\mathrm{tr}(A)$ denotes the trace of matrix A, and m denotes the Lebesgue measure. For a scalar continuous-time signal $w(t)$, we define $|w|_t = \sup_{s\in[0,t]} |w_1(s)|$. Finally, to formulate the stability properties, for non-negative constant d we define

$$\begin{aligned}
\mathcal{K}(d) &:= \{f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | f \text{ continuous}, \\
&\qquad \text{strictly increasing, and } f(0) = d\}, \\
\mathcal{K}_\infty(d) &:= \{f \in \mathcal{K}(d) | f \text{ unbounded}\}, \\
\mathcal{K}^2_\infty &:= \{f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | \\
&\qquad \forall t \geq 0, f(.,t) \in \mathcal{K}_\infty(0), \text{ and } \forall r \geq 0\ f(r,.) \in \mathcal{K}_\infty(0)\} \\
\mathcal{L} &:= \{f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | f \text{ continuous}, \\
&\qquad \text{strictly decreasing, and } \lim_{s\to\infty} f(s) = 0\}, \\
\mathcal{KL} &:= \{f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | f \text{ continuous}, \\
&\qquad \forall t \geq 0, f(.,t) \in \mathcal{K}(0), \text{ and } \forall r > 0\ f(r,.) \in \mathcal{L}\}.
\end{aligned}$$

communication constraints, only a quantized version of it can be encoded in the packet $g(t_s)$. •

At the controller, the estimated state is represented by $\hat{x}$ and evolves during the inter-reception times as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad t \in (t_c^k, t_c^{k+1}), \tag{4}$$

starting from $\hat{x}(t_c^{k+})$, which represents the state estimate of the controller with the information received up to time t_c^k with initial condition $\hat{x}(0)$ (the exact way to construct $\hat{x}(t_c^{k+})$ is explained later in Section 3).

Assumption 2 *The sensor can compute $\hat{x}(t)$ for all time $t \geq 0$.*

Remark 3 We show in Section 4.1 that Assumption 2 is valid for our controller design, provided the sensor knows $\hat{x}(0)$ and the times the actuator performs the control action. In practice, this corresponds to assuming an instantaneous acknowledgment from the actuator to the sensor via the control input, known as *communication through the control input* [Sahai and Mitter, 2006, Matveev and Savkin, 2009]. To obtain such causal knowledge, one can monitor the output of the actuator provided that the control input changes at each reception time. In case the sensor has only access to the plant state, since the system disturbance is bounded (2), assuming that the control input is continuous during inter-reception times and jumps in the reception times such that $B|u(t_c^-) - u(t_c)| > M$, the controller can signal the reception time of the packet to the sensor via $\dot{x}(t)$ (other specific construction are provided in [Ling, 2018, Tatikonda and Mitter, 2004b]). Finally, we note that any necessary condition on the information transmission rate obtained with Assumption 2 in place remains necessary without it too. •

Under Assumption 2, the *state estimation error* at the sensor is

$$z(t) = x(t) - \hat{x}(t), \tag{5}$$

and we rely on this error to determine when a triggering event occurs in our controller design. We next define a modified version of input-to-state practical stability (ISpS) [Jiang et al., 1994, Sharon and Liberzon, 2012], which is suitable for our event-triggering setup with unknown but bounded delay.

Definition 4 *The plant* (1) *is ISpS if there exist $\xi \in \mathcal{KL}$, $\psi \in \mathcal{K}_\infty(0)$, $d \in \mathbb{R}_{\geq 0}$, $\iota \in \mathcal{K}_\infty(d)$, and $\vartheta \in \mathcal{K}_\infty^2$ such that*

$$|x(t)| \leq \xi\left(|x(0)|, t\right) + \psi\left(|w|_t\right) + \iota(\gamma) + \vartheta(|w|_t, \gamma), \ \forall t \geq 0.$$

Note that, for a fixed γ, this definition reduces to the standard notion of ISpS. Given that the initial condition, delay, and system disturbances are bounded, ISpS implies that the state must be bounded at all times beyond a fixed horizon. Our objective is to ensure the dynamics (1) is ISpS given the constraints posed by the system model of Figure 1. Let $b_s(t)$ be the number of bits transmitted in the data payload by the sensor up to time t. The *information transmission rate* is

$$R_s = \limsup_{t\to\infty}(b_s(t)/t) = \limsup_{N\to\infty}\Big(\sum_{k=1}^{N} g(t_s^k) \Big/ \sum_{k=1}^{N}\Delta'_k\Big), \tag{6}$$

where the latter equality follows by noting that, at each triggering time t_s^k, the sensor transmits $g(t_s^k)$ bits. In addition to the data payload, the reception time of the packets carries information. Consequently, let $b_c(t)$ be the amount of information measured in bits included in data payload and timing information received at the controller until time t. The *information access rate* is $R_c = \limsup_{t\to\infty}(b_c(t)/t)$.

Remark 5 We do not consider the bounded delays (3) to be chosen from any specific distribution. Thus, the information that can be gained about the triggering time t_s from the reception time t_c may be quantified by the Rényi 0th-order information functional I_0 [Nair, 2013, Shingin and Ohta, 2012]. Assuming the controller has received N packet by time t, we deduce $b_c(t) = \sum_{k=1}^{N}\left(g(t_s^k) + I_0(t_s^k; t_c^k)\right)$. •

According to the data-rate theorem, if $R_c < A/\ln 2$, the value of the state in (1) becomes unbounded as $t \to \infty$ (the result for plants evolving in continuous time stated in [Hespanha et al., 2002, Theorem 1] does not consider disturbances, but can readily be generalized to account for them), and hence (1) is not ISpS. The data-rate theorem characterizes what is needed by the controller, and does not depend on the specific feedback structure (including aspects such as information pattern at the sensor/controller, communication delays, and whether transmission times are state-dependent, as in event-triggered control, or periodic, as in time-triggered control). In our discussion below, the bound $R_c = A/\ln 2$ serves as a baseline for our results on the information transmission rate R_s to understand the amount of timing information contained in event-triggered control designs in the presence of unknown communication delays.

We do not consider delays, plant disturbances, and initial condition to be chosen from any specific distribution. Therefore, our results are valid for any arbitrary delay, plant disturbances, and initial condition with finite support. In particular, our goal is to find upper and lower bounds on R_s, where the *lower bound* is necessary at least for *a realization* of the initial condition, delay, and disturbances, and the *upper bound* is sufficient for *all realizations* of the initial condition, delay, and disturbances. In addition, our lower bound is necessary for any control policy $u(t)$ to render the plant (1) ISpS under the class of event-triggering strategies described next.

3 Event-triggered design

Here we introduce the class of event-triggered policies considered in this paper to select transmission times that make the plant (1) ISpS. Consider the following class of triggers: for $J \in \mathbb{R}$ positive, the sensor sends a message to the controller at t_s^{k+1} if

$$|z(t_s^{k+1})| = J, \tag{7}$$

provided $t_c^k \leq t_s^{k+1}$ for $k \in \mathbb{N}$ and $t_s^1 \geq 0$. A new transmission happens only after the previous packet has been received by the controller. Since the triggering time t_s is a real number, its knowledge can reveal an unbounded amount of information to the controller. However, due to the unknown delay in the channel, the controller does not have perfect knowledge of it. In fact, both the finite data rate and the delay mean that the controller may not be able to compute the exact value of $x(t_c)$. To address this, let $\bar{z}(t_c)$ be an estimated version of $z(t_c)$ reconstructed by the controller knowing $|z(t_s)| = J$, the bound (3) on the delay, and the packet received through the channel. Using $\bar{z}(t_c)$, the controller updates the state estimate via the *jump strategy*,

$$\hat{x}(t_c^+) = \bar{z}(t_c) + \hat{x}(t_c). \tag{8}$$

Note that $|z(t_c^+)| = |x(t_c) - \hat{x}(t_c^+)| = |z(t_c) - \bar{z}(t_c)|$.

We assume the packet size $g(t_s)$ calculated at the sensor is so that

$$|z(t_c^+)| = |z(t_c) - \bar{z}(t_c)| \leq J, \tag{9}$$

is satisfied for all $t_c \in [t_s, t_s + \gamma]$. This property plays a critical role in our forthcoming developments. In particular, we will show later that our controller design for the sufficient characterization on the transmission rate builds on identifying a particular encoding-decoding strategy and a packet size to make (9) hold true. Likewise, our necessary characterization builds on identifying the minimal packet sizes necessary to ensure (9).

The importance of (9) starts to become apparent in the following result: if this inequality holds at each reception time, the state estimation error (5) is bounded for all time.

Lemma 6 *Consider the plant-sensor-channel-controller model with plant dynamics* (1)*, estimator dynamics* (4)*, triggering strategy* (7)*, and jump strategy* (8)*. Assume $|z(0)| = |x(0) - \hat{x}(0)| < J$ and* (9) *holds at all reception times $\{t_c^k\}_{k\in\mathbb{N}}$. Then, for all $t \geq 0$,*

$$|z(t)| \leq Je^{A\gamma} + \frac{|w|_t}{A}\left(e^{A\gamma} - 1\right). \tag{10}$$

PROOF. At the reception time, $z(t_c^{k+})$ satisfies (9), hence using the triggering rule (7), we deduce $|z(t)| \leq J$ for all $t \in (t_c^k, t_s^{k+1}]$. Since J is smaller than the upper bound in (10), and $z(t_c^{(k+1)+})$ satisfies (9), it remains to prove (10) for $t \in (t_s^{k+1}, t_c^{k+1})$. From (1), (4), and (5), we have $\dot{z}(t) = Az(t) + w(t)$ during inter-reception time intervals (t_c^k, t_c^{k+1}). Also, from (7) it follows $(t_s^{k+1}, t_c^{k+1}) \subseteq (t_c^k, t_c^{k+1})$. Thus, for all $t \in (t_s^{k+1}, t_c^{k+1})$, we have

$$z(t) = e^{A(t - t_s^{k+1})} z(t_s^{k+1}) + \int_{t_s^{k+1}}^{t} e^{A(t-\tau)} w(\tau) d\tau. \tag{11}$$

When a triggering occurs $|z(t_s^{k+1})| = J$, hence the absolute value of the first addend in (11) is upper bounded by $Je^{A(t - t_s^{k+1})}$. Also, for the second addend in (11) we have

$$\begin{aligned} &|\int_{t_s^{k+1}}^{t} e^{A(t-\tau)} w(\tau) d\tau| \\ &\leq |w|_t \int_{t_s^{k+1}}^{t} |e^{A(t-\tau)}| d\tau = \frac{|w|_t}{A}\left(e^{A(t-t_s^{k+1})} - 1\right). \end{aligned} \tag{12}$$

By (3), we have $t - t_s^{k+1} \leq t_c^{k+1} - t_s^{k+1} \leq \gamma$, and the result follows. $\square$

Using (2), we deduce from Lemma 6 that $|z(t)| \leq Je^{A\gamma} + \frac{M}{A}\left(e^{A\gamma} - 1\right)$ for all $t \geq 0$. Next, we rule out Zeno behavior (an infinite amount of triggering events in a finite time interval) for our our event-triggered control design. To do this, let $0 < \rho_0 < 1$ be a design parameter, and assume the packet size $g(t_s)$ is selected at the sensor to ensure a stronger version of (9),

$$|z(t_c^+)| = |z(t_c) - \bar{z}(t_c)| \leq \rho_0 J. \tag{13}$$

Clearly, (13) implies (9). The following result shows that given (13) the time between consecutive triggers is uniformly lower bounded.

Lemma 7 *Consider the plant-sensor-channel-controller model with plant dynamics* (1)*, estimator dynamics* (4)*, triggering strategy* (7)*, and jump strategy* (8)*. Assume $|z(0)| = |x(0) - \hat{x}(0)| < J$ and* (13) *holds at all reception times $\{t_c^k\}_{k\in\mathbb{N}}$. Then for all $k \in \mathbb{N}$ $t_s^{k+1} - t_s^k \geq \ln\left(\frac{JA+M}{\rho_0 JA+M}\right)\Big/A$.*

PROOF. By considering two successive triggering times t_s^k and t_s^{k+1} and the reception time t_c^k, from (7) it follows $t_s^k \leq t_c^k \leq t_s^{k+1}$. From (1), (4), and (5), we have $\dot{z}(t) = Az(t) + w(t)$ during inter-reception time intervals (t_c^k, t_c^{k+1}), consequently using the definition of the triggering time t_s^{k+1} (7) it follows $|z(t_c^{k+})e^{A(t_s^{k+1} - t_c^k)}| + |\int_{t_c^k}^{t_s^{k+1}} e^{A(t_s^{k+1}-\tau)} w(\tau) d\tau| \geq J$. Using (13) and (12), we have $\rho_0 Je^{A(t_s^{k+1} - t_c^k)} + (M/A)(e^{A(t_s^{k+1}-t_c^k)} - 1) \geq J$, which is equivalent to $t_s^{k+1} - t_c^k \geq \frac{1}{A}\ln(\frac{J + \frac{M}{A}}{\rho_0 J + \frac{M}{A}})$. The result follows from using $t_s^k \leq t_c^k$ in this inequality. $\square$

Given the uniform lower bound on the inter-event time obtained in Lemma 7, we deduce that the event-triggered control design does not exhibit Zeno behavior. The frequency with which transmission events are triggered is captured by the *triggering rate*

$$R_{tr} = \limsup_{N\to\infty}\left(N\Big/\sum_{k=1}^{N}\Delta'_k\right). \tag{14}$$

Using Lemma 7, we deduce that the triggering rate (14) is uniformly upper bounded under the event-triggered control design, i.e., for *all* initial conditions, possible delay and plant noise values, we have

$$R_{tr} \leq A\Big/\ln\left(\frac{JA + MA}{\rho_0 JA + M}\right). \tag{15}$$

4 Sufficient and necessary conditions on the information transmission rate

Here we derive sufficient and necessary conditions on the information transmission rate (6) to ensure (1) is ISpS. As mentioned above, our approach is based on the characterization of the transmission rate required to ensure that (9) holds at all reception times. Section 4.1 introduces a quantization policy that, together with the event-triggered scheme, provides a complete control design to guarantee (1) is ISpS and rules out Zeno behavior. Section 4.2 presents lower bounds on the packet size and triggering rate required to guarantee (1) is ISpS, leading to our bound on the necessary information transmission rate. We conclude the section by comparing the sufficient and necessary bounds, and discussing their gap.

4.1 Sufficient information transmission rate

We start by showing that, if (9) holds at each reception time $\{t_c^k\}_{k\in\mathbb{N}}$, then a linear controller renders the plant (1) ISpS. We note that similar results exist in the literature (e.g., [Peralez et al., 2018, Heemels et al., 2012a,b, Girard, 2014]) and we here extend them to our event-triggering setup with quantization and unknown delays.

Proposition 8 *Under the assumptions of Lemma 6, the controller $u(t) = -K\hat{x}(t)$ renders* (1) *ISpS, provided $A - BK < 0$.*

PROOF. By letting $u(t) = -K(x(t) - z(t))$, we rewrite (1) as $\dot{x}(t) = (A - BK)x(t) + BKz(t) + w(t)$. Consequently, we have

$$|x(t)| \le e^{(A-BK)t}|x(0)| \qquad (16)$$
$$+ e^{(A-BK)t} \int_0^t e^{-(A-BK)\tau}(BK|z(\tau)| + |w(\tau)|)d\tau.$$

since $A - BK < 0$, the first summand in (16) is a $\mathcal{KL}$ function of $|x(0)|$ and time. Thus, it remains to prove the second summand in (16) is upper bounded by summation of a $\mathcal{K}_\infty(0)$ function of $|w|_t$, a $\mathcal{K}_\infty(d)$ function of γ, and a $\mathcal{K}^2_\infty$ function of $|w|_t$ and γ. The second summand in (16) is upper bounded by $-(1 - e^{(A-BK)t})(BK|z|_t + |w|_t)/(A - BK)$. Since $1 - e^{(A-BK)t} < 1$, using Lemma 6 we deduce the second summand in (16) is upper bounded by $\psi(|w|_t) + \iota(\gamma) + \vartheta(|w|_t, \gamma)$, where $\psi(|w|_t) = (|w|_t / -(A - BK))$ which is a $\mathcal{K}_\infty(0)$ function of $|w|_t$, $\iota(\gamma) = ((BKJe^{A\gamma})/-(A-BK))$ which is a $\mathcal{K}_\infty(d)$ function of γ with $d = \iota(0)$, and $\vartheta(|w|_t, \gamma) = ((BK|w|_t)/-A(A-BK))(e^{A\gamma}-1)$ which is a $\mathcal{K}^2_\infty$ function of γ and $|w|_t$. □

4.1.1 Design of quantization policy

The result in Proposition 8 justifies our strategy to obtain a sufficient condition on the transmission rate to guarantee (1) is ISpS, which consists of finding conditions to achieve (9) for all reception times. Here we specify a quantization policy and determine the resulting estimation error as a function of the number of bits transmitted. This allows us to determine the packet size that ensures (13) (and consequently (9)) holds, thereby leading to a complete control design which ensures (1) is ISpS and rules out Zeno behavior. In turn, this also yields a sufficient condition on the information transmission rate. In our sufficient design the controller estimates $z(t_c)$ as

$$\bar{z}(t_c) = \text{sign}(z(t_s))Je^{A(t_c - q(t_s))}, \qquad (17)$$

where $q(t_s)$ is an estimation of the triggering time t_s constructed at the controller as described next. According to (7), at every triggering event, the sensor encodes t_s and transmits a packet $p(t_s)$. The packet $p(t_s)$ consists of $g(t_s)$ bits of information and is generated according to the following quantization policy. The first bit $p(t_s)[1]$ denotes the sign of $z(t_s)$. As shown in Figure 2,

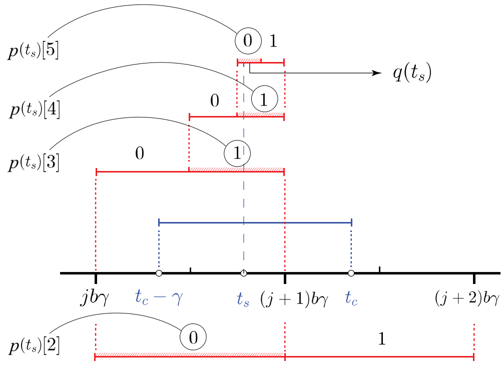

Fig. 2. The encoding-decoding algorithms in the proposed event-triggered control scheme. In this example, we assume $g(t_s) = 5$ and j is an even natural number. The packet $p(t_s)$ of length 5 can be generated and sent to the controller. Recall that $p(t_s)[1]$ encode the sign of $z(t_s)$. After reception and decoding the controller choose the center of the smallest sub-interval as its estimation of t_s, denoted by $q(t_s)$.

the reception time t_c provides information to the controller that t_s could fall anywhere between $t_c - \gamma$ and t_c. Let $b > 1$. To determine the time interval of the triggering event, we break the positive time line into intervals of length $b\gamma$. Consequently, t_s falls into $[jb\gamma, (j+1)b\gamma]$ or $[(j+1)b\gamma, (j+2)b\gamma]$, with j being a natural number. We use the second bit of the packet to determine the correct interval of t_s. This bit is zero if the nearest integer less than or equal to the beginning number of the interval is an even number and is 1 otherwise. This can be written mathematically as $p(t_s)[2] = \text{mod}\big(\lfloor \frac{t_s}{b\gamma} \rfloor, 2\big)$. For the remaining bits of the packet, the encoder breaks the interval containing t_s into $2^{g(t_s)-2}$ equal sub-intervals. Once the packet is complete, it is transmitted to the controller, where it is decoded and the center point of the smallest sub-interval is selected as the best estimate of t_s. Thus,

$$|t_s - q(t_s)| \le b\gamma/2^{g(t_s)-1}. \qquad (18)$$

We have employed this quantization policy in our previous work [Khojasteh et al., 2019] and analyzed its behavior in the case with no disturbances. Next, we extend our analysis to scenarios with both unknown delays and plant disturbances. As discussed in Remark 3, we start by showing that under the proposed encoding-decoding scheme, provided the sensor knows $\hat{x}(0)$ and has causal knowledge of the delay (i.e., the controller acknowledges the packet reception times), then Assumption 2 holds. The proof of the next result is in Appendix A.

Proposition 9 *Under the assumptions of Lemma 7, using the estimation* (17) *and the quantization policy described in Figure 2, if the sensor knows* $\hat{x}(0)$ *and has causal knowledge of delay, then it can calculate* $\hat{x}(t)$ *for all time* $t \ge 0$.

4.1.2 Sufficient packet size

Our next result bounds the difference $|t_s - q(t_s)|$ between the triggering time and its quantized version so that (13) holds at all reception times.

Lemma 10 *Consider the plant-sensor-channel-controller model with plant dynamics* (1)*, estimator dynamics* (4)*, triggering strategy* (7)*, and jump strategy* (8)*. Assume* $|z(0)| = |x(0) - \hat{x}(0)| < J$ *Using the estimation* (17) *and the quantization policy described in Figure 2, if* $|t_s - q(t_s)| \le \frac{1}{A}\ln(1 + \frac{\rho_0 - \frac{M}{JA}(e^{A\gamma}-1)}{e^{A\gamma}})$*, then* (13) *holds for all reception times* $\{t^k_c\}_{k\in\mathbb{N}}$ *if* $J > \frac{M}{A\rho_0}(e^{A\gamma} - 1)$.

PROOF. Using (11), (17), and the triangular inequality, we deduce $|z(t_c) - \bar{z}(t_c)| \le Je^{A(t_c - t_s)}|(1 - e^{A(t_s - q(t_s))})| + |\int_{t_s}^{t_c} e^{A(t_c - \tau)}w(\tau)d\tau|$. By applying the bounds (3), (2), and (12) on first and second addend respectively it follows $|z(t_c) - \bar{z}(t_c)| \le |Je^{A\gamma}(1 - e^{A(t_s - q(t_s))})| + (M/A)\left(e^{A\gamma} - 1\right)$. Therefore, ensuring (13) reduce to

$$|1 - e^{A(t_s - q(t_s))}| \le \eta, \qquad (19)$$

where $\eta = e^{-A\gamma}(\rho_0 - \frac{M}{AJ}(e^{A\gamma} - 1))$. Since $J > \frac{M}{A\rho_0}(e^{A\gamma} - 1)$, we have $0 \le \eta < 1$. Consequently, using (19), we deduce $\ln(1 - \eta)/A \le t_s - q(t_s) \le \ln(\eta + 1)/A$. It follows that to satisfy (13) for all delay values, requiring $|t_s - q(t_s)| \le \min\{|\ln(1 - \eta)|/A, \ln(1 + \eta)/A\}$ suffices, and the result now follows. □

The next result provides a lower bound on the packet size so that (13) is ensured at all reception times.

Theorem 11 *Consider the plant-sensor-channel-controller* *model with plant dynamics* (1)*, estimator dynamics* (4)*, triggering strategy* (7)*, and jump strategy* (8)*. Assume* $|z(0)| = |x(0) - \hat{x}(0)| < J$*, Then there exists a quantization policy that achieves* (13) *for all reception times* $\{t_c^k\}_{k\in\mathbb{N}}$ *with any packet size*

$$g(t_s^k) \geq \max\left\{0, 1 + \log \frac{Ab\gamma}{\ln(1 + \frac{\rho_0 - (M/(JA))(e^{A\gamma}-1)}{e^{A\gamma}})}\right\} \quad (20)$$

where $b > 1$ *and* $J > \frac{M}{A\rho_0}(e^{A\gamma} - 1)$*.*

The proof is a direct consequence of (18) and Lemma 10. The combination of the upper bound (15) obtained for the triggering rate and Theorem 11 yields a sufficient bound on the information transmission rate. To sum it up, we conclude that there exist a information transmission rate

$$R_s \leq \frac{A}{\ln(\frac{JA+M}{\rho_0 JA+M})} \max\left\{0, 1 + \log \frac{Ab\gamma}{\ln(1 + \frac{\rho_0 - (M/(JA))(e^{A\gamma}-1)}{e^{A\gamma}})}\right\}, \quad (21)$$

that is sufficient to ensure (13) and, as a consequence (9), for all reception times $\{t_c^k\}_{k\in\mathbb{N}}$. Therefore, from Proposition 8, the bound (21) is sufficient to ensure the plant (1) is ISpS.

Remark 12 The lower bound given on the packet size in (20) might not be a natural number or might even be zero. If $g(t_s) = 0$, this means that there is no need to put any data payload in the packet and the plant can be stabilized using only timing information. However, in this case, the sensor still needs to inform the controller about the occurrence of a triggering event. Thus, when $g(t_s) = 0$ is sufficient, the sensor can stabilize the system by transmitting a fixed symbol from a unitary alphabet (see Khojasteh et al. [2018a]). In practice, the packet size should be a natural number or zero, so if we do not want to use the fixed symbol from a unitary alphabet, the packet size

$$g(t_s) = \max\left\{1, \left\lceil 1 + \log \frac{Ab\gamma}{\ln(1 + \frac{\rho_0 - (M/(JA))(e^{A\gamma}-1)}{e^{A\gamma}})} \right\rceil\right\}, \quad (22)$$

is sufficient for stabilization (the latter is the one used in our simulations of Section 6). •

4.2 Necessary information transmission rate

Here, we present a necessary condition on the information transmission rate required by any control policy to render plant (1) ISpS under the class of event-triggering strategies described in Section 3. In Section 4.1, to derive a sufficient bound that guarantees (1) is ISpS, our focus has been on identifying *a* quantization policy that could handle *any* realization of initial condition, delay, and disturbance. Instead, the treatment here switches gears to focus on *any* quantization policy, for which we identify at least *a* realization of initial condition, delay, and disturbance that requires the necessary bound on the information transmission rate.

We start our discussion by making the following observation about the property (9). If this property is not satisfied at an arbitrary reception time t_c^k, i.e., $z(t_c^k) > J$, and $w(t) > 0$ or $w(t) < 0$ for all $t \geq t_c^k$, then t_c^k will be the last triggering time. In this case, after t_c^k, the controller needs to estimate the inherently unstable plant in open loop. In this case, there exists a realization of the initial condition, system disturbances, and delay for which the absolute value of the state estimation error grows exponentially with time. Thus, for any given control policy, there exists a realization for which the absolute value of the state tends to infinity with time and (1) is not ISpS.

As a consequence of this observation, our strategy to provide a necessary condition for (1) to be ISpS consists of identifying a necessary condition on the information transmission rate R_s to have (9) at all reception times $\{t_c^k\}_{k\in\mathbb{N}}$. In turn, we do this by finding lower bounds on the packet size $g(t_s)$ and the triggering rate R_{tr}. We do this in two steps: first, we find a lower bound on the number of bits transmitted at each triggering event which holds irrespective of the triggering rate. Then, we find a lower bound on the triggering rate, and the combination leads us to the necessary condition on R_s.

4.2.1 Necessary packet size

We rely on (11) to define the uncertainty set of the sensor about the estimation error at the controller $z(t_c)$ given t_s as follows

$$\Omega(z(t_c)|t_s) = \{y : y = \pm J e^{A(t_r - t_s)} + \int_{t_s}^{t_r} e^{A(t_r-\tau)} w(\tau) d\tau, \quad t_r \in [t_s, t_s + \gamma],\ |w(\tau)| \leq M \text{ for } \tau \in [t_s, t_r]\}.$$

Additionally, we define the uncertainty of the controller about $z(t_c)$ given t_c, as follows

$$\Omega(z(t_c)|t_c) = \{y : y = \pm J e^{A(t_c - t_r)} + \int_{t_r}^{t_c} e^{A(t_c-\tau)} w(\tau) d\tau, \quad t_r \in [t_c - \gamma, t_c],\ |w(\tau)| \leq M \text{ for } \tau \in [t_r, t_c]\}.$$

We next show the relationship between these uncertainty sets.

Lemma 13 *Assume the plant-sensor-channel-controller model described in Section 2, with plant dynamics* (1)*, estimator dynamics* (4)*, triggering strategy* (7)*, and jump strategy* (8)*. Moreover, assume* $M \leq AJ$*. Then* $\Omega(z(t_c)|t_s) = \Omega(z(t_c)|t_c)$ *and* $m(\Omega(z(t_c)|t_c)) = 2(M/A + J)(e^{A\gamma} - 1)$*.*

PROOF. Due to symmetry, it is not difficult to show that $\Omega(z(t_c)|t_s)$ is the same as $\Omega(z(t_c)|t_c)$. We characterize the set $\Omega(z(t_c)|t_s)$ as follows. We reason for the case when $z(t_s) = J$ (the argument for the case $z(t_s) = -J$ is analogous). Clearly, $z(t_c)$ takes its largest value when $t_c = t_s + \gamma$ and $w(\tau) = M$ for $\tau \in [t_s, t_c]$, which is equal to $z(t_c) = Je^{A\gamma} + (M/A)(e^{A\gamma} - 1)$. On the other hand, finding the smallest value of $z(t_c)$ is more challenging. First, when $t_c = t_s$ we have

$$z(t_c) = J. \quad (23)$$

Second, by setting $w(\tau) = -M$ for $\tau \in [t_s, t_c]$ and $t_c = t_s + \Delta$,

$$z(t_c) = Je^{A\Delta} - (M/A)(e^{A\Delta} - 1). \quad (24)$$

Taking the derivative of (24) with respect to Δ results in

$$dz(t_c)/d\Delta = AJe^{A\Delta} - Me^{A\Delta} = e^{A\Delta}(AJ - M). \quad (25)$$

If $M \leq AJ$ and the derivative in (25) is non-negative, $z(t_c)$ in (24) would be a non-decreasing function of Δ. Hence, the smallest value of $z(t_c)$ in (24) occurs for $\Delta = 0$ which is equal to the value of $z(t_c)$ in (23). Hence, $\Omega(z(t_c)|t_s) = [J, Je^{A\gamma} + (M/A)(e^{A\gamma} - 1)]$, and the result follows. □

Lemma 13 allows us to find a lower bound on the packet size $g(t_s)$ which is valid irrespective of the triggering rate.

Lemma 14 *Under the assumptions of Lemma 13, if* (9) *holds for all reception times* $\{t_c^k\}_{k\in\mathbb{N}}$*, then the packet size at every triggering event must satisfy*

$$g(t_s^k) \geq \max\left\{0, \log\left((M/(AJ)+1)\left(e^{A\gamma}-1\right)\right)\right\}. \quad (26)$$

PROOF. To ensure (9) for all reception times, we calculate a lower bound on the number of bits to be transmitted to ensure the sensor uncertainty set $\Omega(z(t_c)|t_s)$ is covered by quantization cells of measure $2J$. Therefore, we have $g(t_s) \geq \max\{0, \log(m(\Omega(z(t_c)|t_s))/m(\mathcal{B}(J)))\}$, where $\mathcal{B}(J)$ is a ball centered at 0 of radius J, and we have incorporated the fact that the packet size $g(t_s)$ must be non-negative. From Lemma 13 we have $\log\frac{m(\Omega(z(t_c)|t_s))}{m(\mathcal{B}(J))} \geq \log\frac{(M/A+J)(e^{A\gamma}-1)}{J}$. □

4.2.2 Lower bound on the triggering rate

Having found a lower bound on the packet size, our next step is to determine a lower bound on the triggering rate.

Lemma 15 *Under the assumptions of Lemma 13, for all the quantization policies which ensure* (9) *at all reception times* $\{t_c^k\}_{k\in\mathbb{N}}$*, if there exists a delay realization* $\{\Delta_k \leq \alpha\}_{k\in\mathbb{N}}$*, a disturbance realization, and an initial condition such that*

$$|z(t_c^{k+})| = |z(t_c^k) - \bar{z}(t_c^k)| \geq \Upsilon, \quad (27)$$

for all $k \in \mathbb{N}$*, then there exists a delay realization, a disturbance realization, and an initial condition such that*

$$R_{tr} \geq A\left(\ln\left(e^{A\alpha}(JA+M)/(\Upsilon A+M)\right)\right)^{-1}. \quad (28)$$

PROOF. Using the definition of the triggering time (7), (27), $t_c^k = t_s^k + \Delta_k$, and (11), we have $\Upsilon e^{A(t_s^{k+1}-t_s^k-\Delta_k)} + (M/A)\left(e^{A(t_s^{k+1}-t_s^k-\Delta_k)}-1\right) \leq J$, which is equivalent to

$$e^{A(t_s^{k+1}-t_s^k)} \leq e^{A\Delta_k}(JA+M)/(\Upsilon A+M). \quad (29)$$

By hypothesis, (27) occurs for all $k \in \mathbb{N}$ when $\Delta_k \leq \alpha$. Hence, by (29), we upper bound the triggering intervals as

$$\Delta_k' = t_s^{k+1} - t_s^k \leq A^{-1}\ln\left(e^{A\alpha}(JA+M)/(\Upsilon A+M)\right) (30)$$

The result follows by substituting (30) into (14). □

If we do not limit the collection of permissible quantization policies, a packet may carry an unbounded amount of information, which can bring the state estimation error arbitrarily close to zero at all reception times and for all delay and disturbance values. This would give rise to a conservative lower bound on the transmission rate. Specifically, using $\Delta_k \leq \gamma$, cf. (3), putting $\Upsilon = 0$, and combining (28) and (26) we deduce there exists a delay realization, disturbance realization, and initial condition such that

$$R_s \geq A\frac{\max\left\{0, \log\left(\left(\frac{M}{AJ}+1\right)\left(e^{A\gamma}-1\right)\right)\right\}}{\ln\left(e^{A\gamma}\frac{JA+M}{M}\right)}, \quad (31)$$

is necessary for all quantization policies. To find a tighter necessary condition we instead limit the collection of permissible quantization policies. Since ensuring (9) at each reception time is equivalent to dividing the uncertainty set at the controller $\Omega(z(t_c)|t_c)$ by quantization cells of measure of at most $2J$, our approach is to restrict the class of quantization policies to those that use the minimum possible number of bits to ensure (9).

Assumption 16 *We assume at each triggering time the sensor transmits the smallest possible packet size (data payload) to ensure* (9) *at each reception time for all initial conditions and all possible realizations of the delay and plant disturbance. Moreover, to simplify our analysis in the encoding-decoding scheme, we choose the center of each quantization cell as* $\bar{z}(t_c)$.

Based on this assumption, the sensor brings the uncertainty about $z(t_c)$ at the controller down to a quantization cell of measure at most $2J$, using the smallest possible packet size. The following result, whose proof is in Appendix B, shows that, for this class of quantization policies, there exists a delay realization such that the sensor can only shrink the estimation error for the controller to at most half of the largest value of J dictated by (9).

Lemma 17 *Let* $\beta = \ln\left(1+2AJ/(AJ+M)\right)/A \leq \gamma$*. Under the assumptions of Lemma 13, for all the quantization policies ensuring* (9) *at all reception times* $\{t_c^k\}_{k\in\mathbb{N}}$ *with Assumption 16 in place, there exists a delay realization* $\{\Delta_k \leq \beta\}_{k\in\mathbb{N}}$*, initial condition, and plant disturbance such that*

$$|z(t_c^{k+})| = |z(t_c^k) - \bar{z}(t_c^k)| \geq J/2. \quad (32)$$

Combining Lemmas 15 and 17, we deduce there exists a delay realization, disturbance realization, and initial condition such that

$$R_{tr} \geq A\left(\ln\left(\left(1+\frac{2AJ}{AJ+M}\right)\frac{JA+M}{0.5JA+M}\right)\right)^{-1} \quad (33)$$

is valid for all quantization policies that use the minimum required packet size according to Assumption 16. Finally, the combination of the bounds on the packet size (cf. Lemma 14) and on the triggering rate (cf. (33)) yields the next result.

Theorem 18 *Under the assumptions of Lemma 13, for all the quantization policies which ensure* (9) *at all reception times* $\{t_c^k\}_{k\in\mathbb{N}}$ *with Assumption 16 in place, there exists a delay realization* $\{\Delta_k \leq \beta\}_{k\in\mathbb{N}}$*, a disturbance realization, and an initial condition such that*

$$R_s \geq A\frac{\max\left\{0, \log\left((M/(AJ)+1)\left(e^{A\gamma}-1\right)\right)\right\}}{\ln\left(\left(1+\frac{2AJ}{AJ+M}\right)\frac{JA+M}{0.5JA+M}\right)}. \quad (34)$$

Note that the bound (34) is tighter than the bound in (31). Figure 3 compares our bounds on the sufficient (21) and necessary (34) information transmission rates for (1) to be ISpS. We attribute the gap between them to the fact that, while the necessary condition employs quantization policies with the minimum possible packet size according to Assumption 16, the encoding-decoding scheme proposed in the sufficient design does not generally satisfy this assumption. Also, the fact that we bound the triggering rate and the packet size independently in our analysis might further contribute to the gap.

As depicted in Figure 3, for sufficiently small delay values the timing information is substantial, and the plant can be ISpS in the presence of bounded system disturbances when the sensor transmits data payload at a rate smaller than the one indicated by the data-rate theorem. On the other hand, as the communication delay increases, the timing information becomes less useful and the uncertainty about the state increases at the controller. Since in our design the state estimation error is smaller than the triggering threshold at each reception time (9), for larger values of delay R_s exceeds the access rate prescribed by the data-rate theorem.

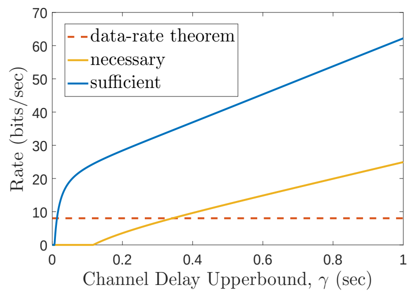

Fig. 3. Illustration of the sufficient (21) and necessary (34) transmission rates as functions of the delay upper bound γ. Here, $A = 5.5651$, $\rho_0 = 0.1$, $b = 1.0001$, $M = 0.4$, and $J = \frac{M}{A\rho_0}(e^{A\gamma} - 1) + 0.1$. The rate dictated by the data-rate theorem is $R_c \geq A/\ln 2 = 8.02874$.

5 Extension to complex linear systems

In this section, we generalize our treatment to complex linear plants with disturbances. The results presented here can be readily applied to multivariate linear plants with disturbance and diagonalizable open loop-gain matrix (possibly, with complex eigenvalues). This corresponds to handling the n-dimensional real plant as n scalar (and possibly complex) plants, and derive a sufficient condition for them. We consider a plant, sensor, communication channel and controller described by the following continuous linear time-invariant system

$$\dot{x} = Ax(t) + Bu(t) + w(t), \tag{35}$$

where $x(t)$ and $u(t)$ belong to $\mathbb{C}$ for $t \in [0, \infty)$. Here $w(t) \in \mathbb{C}$ represents a plant disturbance, which is upper bounded as $\|w(t)\| \leq M$, with $M \in \mathbb{R}_{\geq 0}$. Also, $A \in \mathbb{C}$ with $\text{Re}(A) \geq 0$ (since we are only interested in unstable plants) and $B \in \mathbb{C}$ is nonzero. The model for the communication channel is the same as in Section 2. To establish a baseline for comparison of the bounds on the information transmission rate, we start by stating a generalization of the classical data-rate theorem for the complex plant (35). The proof is in Appendix C.

Theorem 19 *Consider the plant-sensor-channel-controller model with plant dynamics (35). If $x(t)$ remains bounded as $t \to \infty$, then $R_c \geq 2\,\text{Re}(A)/\ln 2$.*

5.1 *Event-triggered control for complex linear systems*

The state estimate $\hat{x}$ evolves according to the dynamics (4) along the inter-reception time intervals starting from $\hat{x}(t_c^{k+})$ with initial condition $\hat{x}(0)$. We use the *state estimation error* defined as (5) with initial condition $z(0) = x(0) - \hat{x}(0)$. A triggering event happens at t_s^{k+1} if

$$\|z(t_s^{k+1})\| = J, \tag{36}$$

provided $t_c^k \leq t_s^{k+1}$ for $k \in \mathbb{N}$ and $t_s^1 \geq 0$, and the triggering radius $J \in \mathbb{R}$ is positive. At each triggering time, the packet $p(t_s)$ of size $g(t_s)$ is transmitted from the sensor to the controller. The packet $p(t_s)$ consists of a quantized version of the phase of $z(t_s)$, denoted $\phi_{q(z(t_s))}$, and a quantized version of the triggering time t_s. By (36), we have $z(t_s) = Je^{i\phi_{z(t_s)}}$. We construct a quantized version, denoted $q(z(t_s))$, of $z(t_s)$ at the controller as $q(z(t_s)) = Je^{i\phi_{q(z(t_s))}}$. Additionally, using the bound (3) and the packet at the controller, the quantized version of t_s is reconstructed and denoted by $q(t_s)$. Hence, at the controller, $z(t_c)$ is estimated as follows

$$\bar{z}(t_c) = e^{A(t_c - q(t_s))} q(z(t_s)). \tag{37}$$

We use the jump strategy (8) to update the value of $\hat{x}(t_c^+)$. Hence, $\|z(t_c^+)\| = \|z(t_c) - \bar{z}(t_c)\|$ holds. At the sensor, the packet size $g(t_s)$ is chosen to be large enough such that

$$\|z(t_c^+)\| = \|z(t_c) - \bar{z}(t_c)\| \leq \rho_0 J, \tag{38}$$

(where $0 < \rho_0 < 1$ is a design parameter) is satisfied for all $t_c \in [t_s, t_s + \gamma]$. Figure 4 shows a typical realization of $z(t)$ under the proposed event-triggered strategy before and after one event. The notion of ISpS remains the same as in Definition 4 by replacing absolute value with complex absolute value.

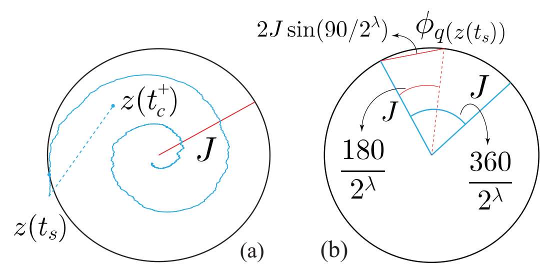

Fig. 4. (a) The blue curve shows the evolution of the state estimation error before and after an event. The trajectory starts with an initial state inside a circle of radius J, and continues spiraling (due to the imaginary part of A) until it hits the triggering threshold radius J. Then it jumps back inside the circle after the update according to (37) and jump strategy (8). During inter-reception time intervals, $\dot{z}(t) = Az(t) + w(t)$, and the observed overshoot beyond the circle is due to the delay in the communication channel. Here, $A = 0.3 + 2i$, $B = 0.2$, $u(t) = -8\hat{x}(t)$, $M = 0.2$, $\gamma = 0.05$ sec, $\rho_0 = 0.9$ and $J = 0.0173$. (b) Estimation of the phase angle after event and transmission of λ bits.

Remark 20 Similarly to Proposition 8, one can show that if (38) occurs at all reception times and (A, B) is a stabilizable pair, then under the control rule $u(t) = -K\hat{x}(t)$, the plant (35) is ISpS, provided the real part of $A - BK$ is negative. As a consequence of this observation, our analysis focues on ensuring (38) at each reception time. The lower bound on the inter-event time of Lemma 7 and the upper bound on the triggering rate (15) also holds replacing A by $\text{Re}(A)$ for the complex plant. •

5.2 *Sufficient information transmission rate*

In this section, we design a quantization policy that, using the event-triggered controller of Section 5.1, ensures the plant (35) is ISpS. We rely on this design to establish a sufficient bound on the information transmission rate.

5.2.1 *Design of quantization policy*

We devote the first λ bits of the packet $p(t_s)$ for quantizing the phase of $z(t_s)$. The proposed encoding algorithm uniformly quantizes the circle into 2^λ pieces of $2\pi/2^\lambda$ radians. After reception, the decoder finds the correct phase quantization cell and selects its center point as $\phi_{q(z(t_s))}$. By letting $\omega = \phi_{z(t_s)} - \phi_{q(z(t_s))}$, as depicted in Figure 4, geometrically we deduce $|\omega| \le \pi/2^{\lambda}$. Furthermore, we use the encoding scheme proposed in Figure 2 to append a quantized version of triggering time t_s of length $g(t_s) - \lambda$ to the packet $p(t_s)$. Hence, we have $p(t_s)[\lambda+1] = \text{mod}\big(\lfloor \frac{t_s}{b\gamma} \rfloor, 2\big)$. For the remaining bits of the packet, the encoder breaks the interval containing t_s into $2^{g(t_s)-\lambda-1}$ equal sub-intervals. Once the packet is complete, it is transmitted to the controller, where it is decoded and the center point of the smallest sub-interval is selected as the best estimate of t_s. Therefore,

$$|t_s - q(t_s)| \le b\gamma/2^{g(t_s)-\lambda}. \tag{39}$$

Note that, given t_s^{k+1}, one can identify $q(t_s^{k+1})$ deterministically. Also, using the first λ bits of the packet, the sensor can find the value of $\phi_{q(z(t_s))}$. Consequently, similar to Proposition 9, if the sensor has a causal knowledge of the delay in the communication channel, it can calculate the state estimation $\hat{x}(t)$ for all time t.

5.2.2 *Sufficient packet size*

Here we show that with a sufficiently large packet size, we can achieve (38) at all reception times $\{t_c^k\}_{k\in\mathbb{N}}$ using the quantization policy designed in Section 5.2.1. The proof of the next result is in Appendix D.

Theorem 21 *Consider the plant-sensor-channel-controller model with plant dynamics* (35)*, estimator dynamics* (4)*, triggering strategy* (36)*, and jump strategy* (8)*. Assume $\|z(0)\| = \|x(0) - \hat{x}(0)\| < J$, then the quantization policy designed above achieves* (38) *for all reception times $\{t_c^k\}_{k\in\mathbb{N}}$ with any packet size lower bounded by*

$$g(t_s) \ge \bar{g} \triangleq \tag{40}$$

$$\max\left\{0, \lambda + \log \frac{\mathrm{Re}(A) b\gamma}{\ln\left(\frac{1+e^{-\mathrm{Re}(A)\gamma}\left(\rho_0 - \frac{M}{\mathrm{Re}(A)J}\left(e^{\mathrm{Re}(A)\gamma}-1\right)\right)}{2\sin(\pi/2^{\lambda+1})+1+\sqrt{2\zeta}}\right)}\right\},$$

provided $\cos\Big(\mathrm{Im}(A)\big(t_s - q(t_s)\big)\Big) = 1 - \zeta$, $b > 1$,

$$\rho_0 \ge \tag{41a}$$

$$\frac{M}{\mathrm{Re}(A)J}\left(e^{\mathrm{Re}(A)\gamma} - 1\right) + e^{\mathrm{Re}(A)\gamma}\left(2\sin(\pi/2^{\lambda+1}) + \sqrt{2\zeta}\right),$$

$$J \ge \frac{M}{\mathrm{Re}(A)\chi}\left(e^{\mathrm{Re}(A)\gamma} - 1\right), \quad \sqrt{2\zeta}e^{\mathrm{Re}(A)\gamma} \le \chi', \tag{41b}$$

$$\lambda > \log\left(\pi \Big/ \arcsin\left(\frac{1-\chi-\chi'}{2e^{\mathrm{Re}(A)\gamma}}\right)\right) - 1, \tag{41c}$$

where $0 < \chi + \chi' < 1$.

Combining the bound on the triggering rate from Remark 20 with Theorem 21, it follows that there exists an information transmission rate with

$$R_s \le \mathrm{Re}(A)\bar{g} \Big/ \ln\left(\frac{J\,\mathrm{Re}(A) + M}{\rho_0 J\,\mathrm{Re}(A) + M}\right), \tag{42}$$

that achieves (38) for all reception times $\{t_c^k\}_{k\in\mathbb{N}}$, and is therefore, sufficient to ensure (35) is ISpS. Figure 5 shows the sufficient information transmission rate in (42) as a function of the upper bound γ on the channel delay. One can observe that for small values of the delay, the sufficient information transmission rate is smaller than the rate required by the data-rate result in Theorem 19, and as the delay upper bound γ increases, the sufficient information transmission rate increases accordingly.

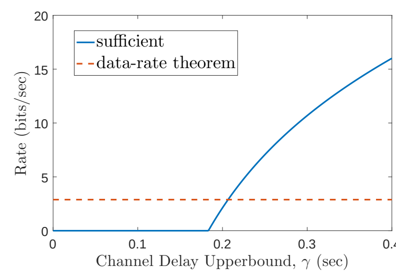

Fig. 5. Sufficient information transmission rate (42) as a function of channel delay upper bound γ. Here $A = 1 + i$, $B = 0.5$, $M = 0.1$, $\rho_0 = 0.9$ and $b = 1.0001$. Also $\lambda = \log\left(\pi/2\arcsin(\frac{7}{8})e^{\mathrm{Re}(A)\gamma}\right)$ and $J = \frac{8M}{\mathrm{Re}(A)}\left(e^{\mathrm{Re}(A)\gamma} - 1\right) + 0.002$. The rate dictated by the data-rate theorem (cf. Theorem 19) is $2\,\mathrm{Re}(A)/\ln 2 = 2.885$.

Remark 22 Depending on whether the system is real or complex, the corresponding triggering criterion is based on the real or complex absolute value, resp., cf. (7) and (36). The controller needs to approximate the phase at which the state estimation error $z(t_s)$ hits the triggering radius. The real case is a particular case of our complex results, since the phase of $z(t_s)$ is then either 0 or π. Thus, for the real case, in our sufficient design, only the first bits of the packet $p(t_s)$ denote the sign of $z(t_s)$. On the other hand, in the complex case, we devote the first λ bits of the packet $p(t_s)$ for quantizing the phase of $z(t_s)$. By putting $A = \mathrm{Re}(A)$, $\lambda = 1$, and $\mathrm{Im}(A) = 0$ (or $\zeta = 0$), our sufficient condition for complex systems (42), becomes equal to (21) except a factor $1 + \sqrt{2}$, which makes (42) larger than (21). The reason for the difference is (D.4), where we find an upper bound on the estimation error of the phase of $z(t_s)$. In the real case, the controller deduces $z(t_s) = J$ or $z(t_s) = -J$, and the estimation error of the phase of $z(t_s)$ is zero. •

6 Simulations

This section presents simulation results validating the proposed event-triggered control scheme for real-valued plants (the interested reader can find simulations for a complex-valued plant in [Khojasteh et al., 2018b]). While our analysis is for continuous-time plants, we perform the simulations in discrete time with a small sampling time $\delta' > 0$. Thus, the minimum upper bound for the channel delay is equal to two sampling times in the digital environment (this is because a delay of at most one sampling time might occur from the time that triggering occurs to the time that the sensor took a sample from the plant state and another delay of at most one sampling time might occur from the time that the packet is received to the time the control input is applied to the plant). We consider a linearized version of the two-dimensional problem of balancing an inverted pendulum mounted on a cart, where the motion of the pendulum is constrained in a plane and its position can be measured by an angle θ. The inverted pendulum has mass m_1, length l, and moment of inertia I. Also, the pendulum is mounted on top of a cart of mass m_2, constrained to move in y direction. The nonlinear equations governing the motion of the cart and pendulum are $(m_1 + m_2)\ddot{y} + \nu\dot{y} + m_1 l\ddot{\theta}\cos\theta - m_1 l\dot{\theta}^2\sin\theta = F$

and $(I + m_1 l^2)\ddot{\theta} + m_1 g_0 l sin\theta = -m_1 l \ddot{y} cos\theta$, where ν is the damping coefficient between the pendulum and the cart and g_0 is the gravitational acceleration. We define $\theta = \pi$ as the equilibrium position of the pendulum and ϕ as small deviations from θ. We derive the linearized equations of motion using small angle approximation, noting that this linearizion is only valid for sufficiently small values of the delay upper bound γ. Define the state variable $s = [y, \dot{y}, \phi, \dot{\phi}]^T$, where y and $\dot{y}$ are the position and velocity of the cart respectively. Assuming $m_1 = 0.2$ kg, $m_2 = 0.5$ kg, $\nu = 0.1$ N/m/s, $l = 0.3$ m, $I = 0.006$ kg/m^2, one can write the evolution of s in time as

$$\dot{s} = As(t) + Bu(t) + w(t), \tag{43}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6730 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1800 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.8180 \\ 0 \\ 4.5450 \end{bmatrix}.$$

In addition, we add the plant noise $w(t) \in \mathbb{R}^4$ to the linearized plant model, and we assume that all of its elements are upper bounded by M. A simple feedback control law can be derived for (43) as $u = -Ks$, where $K = [-1.00 \ -2.04 \ 20.36 \ 3.93]$. is chosen such that $A - BK$ is Hurwitz.

The eigenvalues of the open-loop gain of the plant A are $e = [0 \ -5.6041 \ -0.1428 \ 5.5651]$. Thus, the open-loop gain of the plant A is diagonalizable (all eigenvalues of A are distinct). Using the eigenvector matrix P, we diagonalize the plant to obtain

$$\dot{\tilde{s}} = \tilde{A}\tilde{s}(t) + \tilde{B}\tilde{u}(t) + \tilde{w}(t), \tag{44}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5.6041 & 0 & 0 \\ 0 & 0 & -0.1428 & 0 \\ 0 & 0 & 0 & 5.5651 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 10.0000 \\ -2.3865 \\ 10.0979 \\ 2.2513 \end{bmatrix},$$

where $\tilde{s}(t) = P^{-1}s(t)$ and $\tilde{w}(t) = P^{-1}w(t)$. Also, $\tilde{u}(t) = -\tilde{K}\tilde{s}(t)$ where $\tilde{K} = KP$. For the first three coordinates of the diagonalized plant in (44) the state estimation $\hat{s}$ at the controller simply constructs as $\dot{\hat{s}}_i = \tilde{A}_i\hat{s}(t) + \tilde{B}_i\tilde{u}(t)$, starting from $\hat{s}_i(0)$ for $i \in \{1, 2, 3\}$, where $\tilde{A}_i$ and $\tilde{B}_i$ denote the i^{th} row of $\tilde{A}$ and $\tilde{B}$. Since the first three eigenvalues of A are non-negative, they are inherently stable. Thus, by the data theorem [Sharon and Liberzon, 2012] there is no need to use the communication channel for them, and since $\tilde{A} - \tilde{B}\tilde{K}$ is Hurwitz, $\tilde{u}(t) = -\tilde{K}\tilde{s}(t)$ renders them ISS with respect to system disturbances. Now we apply Theorem 11 to the fourth mode of the plant, which is unstable, to make the whole plant ISpS. Using the problem formulation in Section 2, the estimated state for the unstable mode $\hat{s}_4$ evolves during the inter-reception times as

$$\dot{\hat{s}}_4(t) = 5.5651\hat{s}_4(t) + 2.2513\tilde{u}(t), \quad t \in (t_c^k, t_c^{k+1}), \tag{45}$$

starting from $\hat{s}_4(t_c^{k+})$ and $\hat{s}_4(0)$. Also, a triggering occurs when $|\tilde{z}_4(t)| = |\tilde{s}_4(t) - \hat{s}_4(t)| = J$, where $|\tilde{z}_4(t)|$ is the estate estimation error for the unstable mode, and assuming the previous packet is already delivered to the controller. In the simulation environment, since the sampling time is small, a triggering happens as soon as $|\tilde{z}_4(t)|$ is equal or greater than J and the previous packet has been recived by the controller. Let $\lambda_4 = 5.5651$ be the eigenvalue corresponding to the unstable mode. By Theorem 11, we choose $J = (M/(\lambda_4\rho_0))(e^{\lambda_4\gamma} - 1) + 0.005$, and the size of the packet for all t_s to be (22), where $b = 1.0001$ and $\rho_0 = 0.9$. We use the packet size given in (22) for the simulations.

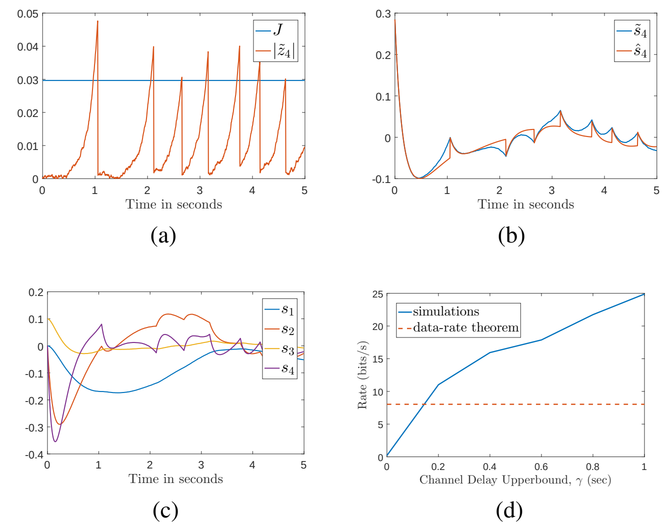

Fig. 6. Simulation results for linearized inverted pendulum on a cart example. (a) shows the evolution of the absolute value of the state estimation error (a) for the unstable mode of the plant in (44). (b) shows the evolution of the unstable state in (44) and its estimate in (45). (c) shows the evolution of all the states in (43). (d) shows the information transmission rate in the simulation as compared to the data-rate theorem. Note that the rate does not start at $\gamma = 0$ because the minimum channel delay upper bound is equal to two sampling time (0.005 seconds in this example). The simulation parameters are $\tilde{s}(0) = P^{-1}[0, 0, 0, 0.1001]^T$, $\hat{s}(0) = P^{-1}[0, 0, 0, 0.10]^T$, simulation time $T = 5$ seconds, and sampling time $\delta' = 0.005$ seconds, For (a)-(c), $\gamma = 0.1$ sec, $g(t_s) = 4$ bits, $M = 0.05$, and in (d) $g(t_s)$ is calculated using (22) with $M = 0.2$.

Figure 6(a) shows the triggering function for $\tilde{s}_4$ in (44) and the absolute value of the state estimation error for the unstable coordinate, that is, $|\tilde{z}_4(t)| = |\tilde{s}_4(t) - \hat{s}_4(t)|$. As soon as the absolute value of this error is equal or greater than the triggering function, the sensor transmits a packet, and the jumping strategy adjusts $\hat{s}_4$ at the reception time to ensure the plant is ISpS. Note that the amount this error exceeds the triggering function depends on the random channel delay upper bounded by γ. Figure 6(b) presents the evolution of the unstable state in (44) and its estimation in (45). Figure 6(c) shows the evolution of all the actual states of the linearized plant (43). Finally, Figure 6(d) presents the simulation of information transmission rate versus the delay upper bound γ in the communication channel for stabilizing the linearized model of the inverted pendulum. It can be seen that for small γ, the plant is ISpS with an information transmission rate smaller than the one prescribed by the data-rate theorem.

7 Conclusions

We have presented an event-triggered control scheme for the stabilization of noisy, scalar real and complex, continuous, linear time-invariant systems over a communication channel subject to random bounded delay. We have developed an algorithm for encoding-decoding the quantized version of the estimated state, leading to the characterization of a sufficient transmission rate for stabilizing these systems. We also identified a necessary condition on the transmission rate for real systems. Future work will study the identification of necessary conditions on the transmission rate in complex systems, develop event-triggered designs for vector systems with real and complex eigenvalues, and perform experiments with the proposed controllers in practical scenarios.

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A Proof of Proposition 9

PROOF. The proof is based on induction. Using $\hat{x}(0)$ sensor can construct the value of $\hat{x}(t)$ for $t \in (0, t_c^1)$ according to (4). Note that we are using the proposed quantizer in Figure 2, hence given t_s^1, $q(t_s^1)$ gets identified deterministically. Consequently, given t_c^1 and using (17), the sensor constructs the value of $z(t_c^{1+})$ and it determines the value of $\hat{x}(t_c^{1+})$.

Now assuming that the sensor is aware of the value of $\hat{x}(t_c^{k+})$ we will prove that the sensor can find the value of $\hat{x}(t_c^{(k+1)+})$ too. Since the sensor is aware of the $\hat{x}(t_c^{k+})$ and it knows that $\hat{x}(t)$ evolves according to (4) for $t \in (t_c^k, t_c^{k+1})$ starting from $\hat{x}(t_c^{k+})$ sensor can calculate all the values of $\hat{x}(t)$ until t_c^{k+1}. Using our proposed quantizer and given t_s^{k+1}, $q(t_s^{k+1})$ can be identified deterministically, therefore by knowing the value of $(k+1)^{th}$ delay the sensor can calculate the value of $\bar{z}(t_c^{(k+1)+})$ from (17). Then using the jump strategy (8) it can calculate $\hat{x}(t_c^{(k+1)+})$. So the result follows. ☐

B Proof of Lemma 17

PROOF. Without loss of generality assume that $z(t_s) = J$ throughout this proof. We also consider the realization of $w(t) = M$ for all time t. We first show β is the time needed for the state estimation error to grow from $z(t_s)$ to $z(t_s) + 2J$. From (11), we deduce at delay β we have

$$z(t_c) = e^{A\beta}J + (M/A)\left(e^{A\beta} - 1\right). \tag{B.1}$$

By combining (B.1), the bound on β, and $z(t_s) = J$ it follows $z(t_c) = z(t_s) + 2J$. Hence, the value of $z(t_c)$ sweeps an area of measure $2J$ when the delay takes values in $[0, \beta]$.

We continue by distinguishing between two classes of quantization cells. We call a quantization cell *perfect*, if its measure is equal to $2J$, and when the measure of a quantization cell is less than $2J$ we call it *defective*. Using these definitions we now prove the occurrence of (32) with delay of at most β, in three different cases. First, when $z(t_s)$ is in a perfect cell, clearly for a delay of at most β we have $|z(t_c^k) - \bar{z}(t_c^k)| \geq J$, and (32) follows. Second, when $z(t_s)$ is in a defective cell which is adjacent to a perfect cell, for a delay of at most β the value of $z(t_c)$ sweeps the area of the defective cell and $z(t_c)$ inters the adjacent perfect cell. Thus, with delay at most β we have $|z(t_c^k) - \bar{z}(t_c^k)| \geq J/2$, where $\bar{z}(t_c^k)$ is the center of the adjacent perfect cell with radius J, and (32) follows. It remains to check the assertion when $z(t_s)$ is in a defective quantization cell which is adjacent to another defective quantization cell. Due to the restriction on the quantization policies as in Assumption 16, the sensor transmits the minimum required bits to divide the uncertainty set at the controller to quantization cell of measure of at most $2J$. If the measure of union of two adjacent cells is at most $2J$, these two balls could be replaced by one quantization cell to reduce the number of quantization cells. As a consequence, under Assumption 16, the measure of union of two adjacent quantization cells is greater than $2J$. Assume the defective quantization cell that contain $z(t_s)$ is of the measure μ_1 and the measure of the adjacent defective cell is μ_2. As a result, we have $\mu_1 + \mu_2 > 2J$. Therefore, at least one of the μ_1 or μ_2 is at least J, thus with a delay of at most β, we have $|z(t_c^k) - \bar{z}(t_c^k)| \geq J/2$, and (32) follows. ☐

C Proof of Theorem 19

PROOF. It is enough to prove the assertion when $w(t) = 0$. By rewriting (35) when $w(t) = 0$ we have $\dot{\mathrm{Re}(x)} + i\dot{\mathrm{Im}(x)} = \mathrm{Re}(A)\,\mathrm{Re}(x) - \mathrm{Im}(A)\,\mathrm{Im}(x) + i(\mathrm{Re}(A)\,\mathrm{Im}(x) + \mathrm{Im}(A)\,\mathrm{Re}(X))$, which is equivalent to

$$\begin{bmatrix} \dot{\mathrm{Re}(x)} \\ \dot{\mathrm{Im}(x)} \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(x)(t) \\ \mathrm{Im}(x)(t) \end{bmatrix}.$$

Since $\|x\| = \sqrt{\mathrm{Re}(x)^2 + \mathrm{Im}(x)^2}$, if $\mathrm{Re}(x)$ or $\mathrm{Im}(x)$ becomes unbounded, $\|x\|$ becomes unbounded. Consequently, using [Hespanha et al., 2002, Theorem 1], we need to have

$$R_c \geq \mathrm{tr}\left(\begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix}\right) / \ln 2. \qquad \square$$

D Proof of Theorem 21

PROOF. In our design, the controller estimates $z(t_c)$ as in (37), and the encoding-decoding scheme is as depicted in Figures 2 and 4. Using (11), (37), and the triangle inequality, it follows

$$\begin{aligned} &\|z(t_c) - \bar{z}(t_c)\| \leq \\ &\left\|\left(e^{A(t_c - t_s)}z(t_s) - e^{A(t_c - q(t_s))}q\left(z(t_s)\right)\right)\right\| \\ &+ \left\|\int_{t_s}^{t_c} e^{A(t_c-\tau)}w(\tau)d\tau\right\|. \end{aligned} \tag{D.1}$$

Similarly to (12), since $\|w(t)\| \leq M$, the second summand in (D.1) is upper bounded as

$$\left\|\int_{t_s}^{t_c} e^{A(t_c-\tau)}w(\tau)d\tau\right\| \leq \frac{M}{\mathrm{Re}(A)}\left(e^{\mathrm{Re}(A)\gamma} - 1\right). \tag{D.2}$$

To find a proper upper bound on the first summand in (D.1), assuming $q\left(z(t_s)\right) = z(t_s) - v_1$ and $q(t_s) = t_s - v_2$, we have

$$\begin{aligned} &\left\|e^{At_c}\left(e^{-At_s}z(t_s) - e^{Aq(t_s)}q\left(z(t_s)\right)\right)\right\| = \\ &\left\|e^{A(t_c-t_s)}\left(z(t_s) - e^{Av_2}\left(z(t_s) - v_1\right)\right)\right\| \leq \\ &e^{\mathrm{Re}(A)\gamma}\left(J\|1 - e^{Av_2}\| + e^{\mathrm{Re}(A)v_2}\|v_1\|\right). \end{aligned} \tag{D.3}$$

Next, we find an upper bound of $\|v_1\|$. Since the sensor devotes λ bits to transmit a quantized version of the phase of $z(t_s)$ to the controller, we have the upper bound $|\omega| \leq \pi/2^\lambda$ on the difference of the phases of $z(t_s)$ and $q(z(t_s))$. Also, over $[-\pi, \pi]$, the cosine function is concave, with global maximum at 0. Hence, as depicted in Figure 4, from the law of cosines, we have

$$\begin{aligned} &\|v_1\| = \|z(t_s) - q\left(z(t_s)\right)\| \leq \\ &\sqrt{2J^2(1 - \cos(\pi/2^\lambda))} = 2J\sin(\pi/2^{\lambda+1}). \end{aligned} \tag{D.4}$$

Combining this with (D.3), the first summand in (D.1) is upper bounded by

$$Je^{\mathrm{Re}(A)\gamma}\left(\|1-e^{Av_2}\| + 2e^{\mathrm{Re}(A)v_2}\sin(\pi/2^{\lambda+1})\right).$$

Note that $\|1-e^{Av_2}\|^2 = (1-e^{\mathrm{Re}(A)v_2})^2 + 2e^{\mathrm{Re}(A)v_2}\zeta$, where $\cos(\mathrm{Im}(A)v_2) = 1-\zeta$, and $0 \le \zeta \le 2$. Thus, the first summand in (D.1) is upper bounded by

$$Je^{\mathrm{Re}(A)\gamma}\Big(|1-e^{\mathrm{Re}(A)v_2}| + \sqrt{2e^{\mathrm{Re}(A)v_2}\zeta} + 2e^{\mathrm{Re}(A)v_2}\sin(\pi/2^{\lambda+1})\Big).$$

For any positive real number ϵ we know $\epsilon + 1/\epsilon \ge 2$, hence, $e^{\mathrm{Re}(A)v_2} - 1 \ge 1 - e^{-\mathrm{Re}(A)v_2}$. Therefore, for the rest of the proof, and without loss of generality, we assume $v_2 \ge 0$, and the first summand in (D.1) is upper bounded by

$$Je^{\mathrm{Re}(A)\gamma}\Big(e^{\mathrm{Re}(A)v_2} - 1 + \sqrt{2\zeta}e^{\mathrm{Re}(A)v_2} + 2e^{\mathrm{Re}(A)v_2}\sin(\pi/2^{\lambda+1})\Big). \tag{D.5}$$

Combining (D.1), (D.2), and (D.5) we deduce

$$e^{\mathrm{Re}(A)v_2} \le \frac{1 + e^{-\mathrm{Re}(A)\gamma}\left(\rho_0 - \frac{M}{\mathrm{Re}(A)J}\left(e^{\mathrm{Re}(A)\gamma}-1\right)\right)}{2\sin(\pi/2^{\lambda+1}) + 1 + \sqrt{2\zeta}} \tag{D.6}$$

which suffices to ensure (38). Recalling $v_2 = t_s - q(t_s)$, using (39) and by setting

$$\frac{b\gamma}{2^{g(t_s)-\lambda}} \le \frac{1}{\mathrm{Re}(A)}\ln\left(\frac{1 + e^{-\mathrm{Re}(A)\gamma}\left(\rho_0 - \frac{M}{\mathrm{Re}(A)J}\left(e^{\mathrm{Re}(A)\gamma}-1\right)\right)}{2\sin(\pi/2^{\lambda+1}) + 1 + \sqrt{2\zeta}}\right),$$

(D.6) is ensured. Consequently, the packet size in (40) is sufficient to ensure (38) for all reception times. However, (D.6) is well defined only when the upper bound in (D.6) is at least one, namely

$$e^{-\mathrm{Re}(A)\gamma}\left(\rho_0 - \frac{M}{\mathrm{Re}(A)J}\left(e^{\mathrm{Re}(A)\gamma}-1\right)\right) \ge 2\sin(\pi/2^{\lambda+1}) + \sqrt{2\zeta},$$

which holds because of (41a). Moreover, the design parameter ρ_0 in (38) should be in the open interval $(0,1)$. Therefore, the lower bound in (41a) should be smaller than 1, namely

$$\frac{M}{\mathrm{Re}(A)J}\left(e^{\mathrm{Re}(A)\gamma}-1\right) + e^{\mathrm{Re}(A)\gamma}(2\sin(\pi/2^{\lambda+1}) + \sqrt{2\zeta}) < 1.$$

The result now follows by noting that (41b), and (41c) ensure this inequality holds. □